The studies of such changes are done by the use of the normal form theory and the Lyapunov–Schmidt reduction method where the eigenvalues of the linearized map move along a circle of radius \((1 - \epsilon)^{1/2}\) in the complex plane. The results are: (i) one set of three-cycle branches, which is hyperbolic, becomes separated from the origin under the dissipative perturbation; (ii) a pair of four-cycle branches, which can be hyperbolic for both branches, or hyperbolic for one branch and stable for the other one depending on the condition, is separated from the origin also under the same perturbation; (iii) one set of \(n\)-cycle \((n \geq 5)\) branches, which is elliptic for the area-preserving case, is retained under the perturbation for sufficiently small \(\epsilon\).

1. Introduction

In this paper, the investigation of the effect of perturbation (\(\epsilon\)-perturbation) on the dynamic behavior of a one-parameter family of two-dimensional area-preserving maps, which is the discrete version of a nonintegrable autonomous Hamiltonian system with two degrees of freedom or nonautonomous system with one degree of freedom, is described. Many studies on the area-preserving maps have been carried out in the past [Mackay & Meiss, 1987; Mackay, 1982]. More recently, J. P. Van der Weele and his coworkers [1988] have examined the bifurcation behaviors of an elliptic fixed point at each resonance point, which they expressed in terms of the squeeze effect. It was further reported that the squeeze effect becomes incomplete for an unstable (hyperbolic) three-cycle in the presence of \(\epsilon\)-perturbation. The methods used for this study was the well-known normal form theory and the Lyapunov–Schmidt reduction method. The idea of the subharmonic bifurcation has already been discussed concisely in Cvitanović [1989] in which he gave a full discussion about the universality of the infinite sequence of \(m/n\) period \(n\)-tupling of complex mapping.

We show that the complete and incomplete squeeze effects can both be explained by extending the techniques used in Kim & Lee [1992] for subharmonic bifurcations in area-preserving maps.

2. Normal Form of a Weakly Dissipative Map

We start with the following form of a two-dimensional map \(F : \mathbb{R}^2 \to \mathbb{R}^2\), which depends on the parameter \(c\) and \(\epsilon \geq 0\) [Van der Weele et al., 1988],

\[
\begin{bmatrix}
  x' \\
  y'
\end{bmatrix} = F(x, y, c, \epsilon) = \begin{bmatrix}
  2c & -1 \\
  1 - \epsilon & 0
\end{bmatrix} \begin{bmatrix}
  x \\
  y
\end{bmatrix} + \begin{bmatrix}
  f(x) \\
  0
\end{bmatrix},
\]

(1)
where \( f(x) = \sum_{k=2}^{\infty} a_k x^k \). It can be seen that, if 
\( \varepsilon = 0 \), (1) becomes an area-preserving map. If \( |c| \geq (1 - \varepsilon)^{1/2} \), the origin is a saddle point, but when \( |c| < (1 - \varepsilon)^{1/2} \), the eigenvalue lies on the circle of radius \( (1 - \varepsilon)^{1/2} \), complex conjugate to each other. Therefore, \((x, y)\) can be transformed to \((\xi, \eta)\) by

\[
P = \begin{bmatrix} 0 & 1 \\ -(1 - \varepsilon)^{1/2} \sin 2\pi(\theta_0 + \mu) & (1 - \varepsilon)^{1/2} \cos 2\pi(\theta_0 + \mu) \end{bmatrix}
\]

In terms of \((\xi, \eta)\), (1) can be written as

\[
\begin{bmatrix} \xi' \\ \eta' \end{bmatrix} = (1 - \varepsilon)^{1/2} \begin{bmatrix} \cos 2\pi(\theta_0 + \mu) & -\sin 2\pi(\theta_0 + \mu) \\ \sin 2\pi(\theta_0 + \mu) & \cos 2\pi(\theta_0 + \mu) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + f(\mu) \begin{bmatrix} \cot 2\pi(\theta_0 + \mu) \\ 1 \end{bmatrix}
\]

where \( c = c(\mu) = \cos 2\pi(\theta_0 + \mu) \) (\( \theta_0 = m/n \) with \( m \) and \( n \) being relatively prime integers). When \( |c| > (1 - \varepsilon)^{1/2} \), the linear transformation by \( P \) is no longer necessary, since the linear part of the map (1) is diagonalizable with real eigenvalues. The case for \( c < -(1 - \varepsilon)^{1/2} \) (period doubling bifurcations) and \( c = 1 \) are discussed in detail in Lichtenberg & Lieberman [1983] and Zisook [1982].

By setting \( z = \xi + i\eta \) and \( \bar{z} = \xi - i\eta \) in (4), we can obtain the two-dimensional map in a complex form

\[
z' = (1 - \varepsilon)^{1/2} \lambda(\mu) z + \frac{\lambda(\mu)}{\Im \lambda(\mu)} f\left( \frac{z - \bar{z}}{2i} \right)
\]

where \( \lambda(\mu) = e^{2\pi i(\theta_0 + \mu)} = 1 + 2\pi i \mu + O(|\mu|^2) \) and \( \Im z \) denotes the imaginary part of a complex number. If we write the last term on the right hand side of (5) in the following form

\[
\frac{\lambda(\mu)}{\Im \lambda(\mu)} f\left( \frac{z - \bar{z}}{2i} \right) = \sum_{p,q} c_{pq}(\mu) z^p \bar{z}^q
\]

the coefficients \( c_{pq}(\mu) \) in (6) are given by

\[
\begin{align*}
c_{20}(\mu) &= -\frac{a_2}{4} \frac{\lambda(\mu)}{\Im \lambda(\mu)}, \\
c_{11}(\mu) &= \frac{a_2}{2} \frac{\lambda(\mu)}{\Im \lambda(\mu)}, \\
c_{02}(\mu) &= -\frac{a_2}{4} \frac{\lambda(\mu)}{\Im \lambda(\mu)}
\end{align*}
\]

The standard procedure for the next step would be to transform (5) into a normal form by the following \( \mu \)-dependent nonlinear change of coordinates

\[
\begin{bmatrix} x \\ y \end{bmatrix} = P \begin{bmatrix} \xi \\ \eta \end{bmatrix}
\]

[Iooss, 1979; Siegel & Moser, 1971]:

\[
z + \psi(\mu, \varepsilon, \omega, \bar{\omega}) \equiv T(\omega),
\]

where \( \psi(\mu, \varepsilon, \omega, \bar{\omega}) = \sum_{l \geq 2} \psi_l(\mu, \varepsilon, \omega, \bar{\omega}) \) with \( \psi_l(\mu, \varepsilon, \omega, \bar{\omega}) = \sum_{p+q=l} c_{pq}(\mu) z^p \bar{z}^q \), \( l \leq 2 \), and simplify the nonlinear terms with suitable choices of \( \psi_{pq} \). In order to do this, (5) is rearranged to the following form:

\[
z' = \lambda(\mu) z + R_1(\mu, z, \bar{z})
\]

where \( \lambda(\mu) = (1 - \varepsilon)^{1/2} \lambda(\mu) \) and \( R_1(\mu, z, \bar{z}) = \sum_{p+q=l} c_{pq}(\mu) z^p \bar{z}^q \). We can then find a \( \mu \)-dependent local diffeomorphism of the form (8) analogous to the area-preserving case which transforms \( R_1(\mu, z, \bar{z}) \) to the forms shown below with suitable choices of \( \psi_{pq} \) for each \( n \):

(i) for \( n = 3 \),

\[
\lambda(\mu) \omega + b_{02}(\mu) \omega^2 + b_{21}(\mu) \omega^3 \omega + O(|\omega|^4)
\]

(ii) for \( n = 4 \),

\[
\lambda(\mu) \omega + b_{21}(\mu) \omega^2 \omega + b_{03}(\mu) \omega^3 + O(|\omega|^5)
\]

(iii) for \( n \geq 5 \),

\[
\lambda(\mu) \omega + b_{21}(\mu) \omega^2 \omega + b_{03}(\mu) \omega^3 \omega^2 + b_{l_p-1}(\mu) \omega^{p-1} + b_{0,n-1}(\mu) \omega^{n-1} + O(|\omega|^n)
\]

where \( 2p - 1 \leq n \) and the coefficients are calculated to be \( b_{02}(0) = c_{02}(0) \), \( b_{21}(0) = c_{21}(0) + |c_{11}(0)|^2 /
(1 - \lambda_0) + 2|c_{02}(0)|^2/(\lambda_0^2 - \lambda_0) + (2\lambda_0 - 1)c_{11}(0)c_{20}(0)/[\lambda_0(1 - \lambda_0)], b_{03}(0) = c_{03}(0) + c_{11}(0)c_{20}(0)/(\lambda_0^2 - \lambda_0) + 2c_{02}(0)c_{20}(0)/(\lambda_0^2 - \lambda_0), \text{ etc., and } \lambda_0 = \lambda(0). \text{ Although the exact resonance condition for the above surviving terms in the area-preserving case is broken by the presence of perturbation in this case, we retain the transformation method to get the analogous normal form, without which the proper limit of } \varepsilon \to 0 \text{ and the well-behaved transformation equation cannot be defined. Thus combining with (9), we get the following normal forms:}

(i) for } n = 3,
\begin{align*}
\omega' &= F^*(\omega) \\
&= \lambda_2(\mu) + c_{02}(\mu)\bar{\omega}^2 + O(|\omega|^3),
\end{align*}
(13)

(ii) for } n = 4,
\begin{align*}
\omega' &= F^*(\omega) \\
&= \lambda_2(\mu) + b_{21}(\mu)\omega^2\bar{\omega} + b_{03}(\mu)\bar{\omega}^3 \\
&\quad + O(|\omega|^4),
\end{align*}
(14)

(iii) for } n \geq 5,
\begin{align*}
\omega' &= F^*(\omega) \\
&= \lambda_2(\mu) + b_{21}(\mu)\omega^2\bar{\omega} + b_{03}(\mu)\bar{\omega}^3 \\
&\quad + b_{p,p-1}(\mu)\omega^p\bar{\omega}^{p-1} + b_{0,n-1}(\mu)\bar{\omega}^{n-1} \\
&\quad + O(|\omega|^n).
\end{align*}
(15)

3. The Lyapunov–Schmidt Method

Suppose there exists a set of } n\text{-periodic points in the map } F^*(\omega) \text{ and let } \omega = (\omega_1, \ldots, \omega_n) \in \mathbb{C}^n \text{ be the coordinates of those } n\text{-periodic points, that is}

\begin{align*}
F^*(\omega_1) &= \omega_2, \\
F^*(\omega_2) &= \omega_3, \ldots, F^*(\omega_n) &= \omega_1.
\end{align*}
(16)

Then (16) can be rewritten as
\begin{equation}
S\omega = F(\omega)
\end{equation}
(17)

where
\begin{equation}
S = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
1 & 0 & 0 & \cdots & 0
\end{bmatrix}
\quad \text{and } F(\omega) = \begin{bmatrix}
F(\omega_1) \\
F(\omega_2) \\
\vdots \\
F(\omega_n)
\end{bmatrix}
\end{equation}
(18)

Note that } S^n = I \text{ and } \lambda_0^n = 1. \text{ By the linear change of coordinates}

\begin{equation}
y = P(x), \text{ where } P = \begin{bmatrix}
1 & 1 & & & \\
1 & \lambda_0 & \lambda_0^{-1} & & \\
& \ddots & \ddots & \ddots & \\
& & 1 & \lambda_0^{-1} & \lambda_0^{-(n-1)^2}
\end{bmatrix}
\end{equation}
(19)

(17) can be rewritten in terms of } y:

\begin{equation}
\Lambda y = P \mathcal{F}(P^{-1}y),
\end{equation}

where
\begin{equation}
\Lambda = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & \bar{\lambda}_0 & \cdots & 0 \\
& \ddots & \ddots & \ddots \\
0 & 0 & \cdots & \bar{\lambda}_0^{n-1}
\end{bmatrix}
\end{equation}

Defining the map } \Phi : \mathbb{C}^n \times \mathbb{R} \to \mathbb{C}^n \text{ by}

\begin{equation}
\Phi(y) = P \mathcal{F}(P^{-1}y), \quad \Lambda y = 0,
\end{equation}

and using (9) in the form of

\begin{equation}
z' = \lambda(\mu)z + R_1(\mu, z, \bar{z}) + \lambda(\mu)R_2(\mu, z),
\end{equation}

where } R_2(\mu, z) \sum_{l=1}^n \frac{(-1)^{l+1}}{2} 1 \cdot 3 \cdot 5 \cdots (2l - 1) \geq 1, \text{ we get}

\begin{equation}
\Phi(y) = (\lambda(\mu)I - \Lambda)y + PR_1(P^{-1}y) \\
+ \lambda(\mu)R_2(\mu, P^{-1}y) \\
+ \mathcal{E}y + PR_1(\mu, P^{-1}y, P^{-1}y) \\
+ \lambda(\mu)R_2(\mu, P^{-1}y),
\end{equation}

and it can be easily seen that the kernel of the linear part of (24) is a one-dimensional subspace of } \mathbb{C}^n;

\begin{equation}
\ker \mathcal{L} = \{y_n v_n | \forall z \in \mathbb{C}, v_n(0, 0, \ldots, 1)^T \in \mathbb{C}^n\}.
\end{equation}

Let } \mathcal{E} : \mathbb{C}^n \to (\ker \mathcal{L})^\perp \text{ be the projection operator. Then } I - \mathcal{E} : \mathbb{C}^n \to \ker \mathcal{L}, \text{ such that } \mathcal{E}y = (y_1, \ldots, y_n, 0)^T = u \text{ and } (I - \mathcal{E})y = y_nv_n. \text{ It can be shown easily that } \mathcal{E}, I - \mathcal{E}, \text{ and } \mathcal{L} \text{ commute with one another.}
Therefore, the equation $\Phi(y) = 0$ becomes equivalent to the following pair of equations:

\begin{align}
\mathcal{E}\Phi(y_n v_n + u) &= 0, \tag{26a} \\
(I - \mathcal{E})\Phi(y_n v_n + u) &= 0. \tag{26b}
\end{align}

The principle underlying the Lyapunov–Schmidt method [Iooss, 1979] is that even though (24) is not solvable, (26a) may be solved for $n - 1$ variables of $y$ by the implicit function theorem. Then (26b) yields an equation for the unknown $y_n$ if the values for these $n - 1$ variables, $y_1, \ldots, y_{n-1}$ are substituted into (26b).

Therefore, if we define the reduced mapping $g$ as

$$g(y_n, \mu, \epsilon) = \langle (I - \mathcal{E})\Phi(y_n v_n + u), v_n \rangle,$$ \hspace{1cm} (27)

the zeros of $g(y_n, \mu, \epsilon)$ are then in one-to-one correspondence with the zeros of $\Phi(y)$. The following properties are useful for the further analysis (proofs are given in Iooss [1979]).

**Property 1**

$$u = u(|\mu| y_n + |y_n|^2 + \epsilon |y_n|).$$ \hspace{1cm} (28)

**Property 2**

$$u(\lambda_0 y_n) = \Lambda u(y_n).$$ \hspace{1cm} (29)

**Property 3.** Let $z = y_n/n$. Then the equation $g(y_n) = 0$ is equivalent to the following equation in $C$:

$$\lambda_0 z = \lambda(z) + R_1(z, \bar{z}) + \lambda(\mu)R_2(\epsilon, z) = \lambda_\epsilon(z) + R_1(z, \bar{z}),$$ \hspace{1cm} (30)

The advantage of these properties is that $n$-dimensional Eq. (27) can be reduced to a one-dimensional equation [Golubitsky et al., 1988], which can then be solved analytically by the implicit function theorem. If there exists $z = y_n/n$ which satisfies (30), then a set of $n$-periodic fixed points ($\omega_1, \ldots, \omega_n$) are given in terms of $y_n$ as

$$\omega_k = (P^{-1}y)_k = \frac{1}{n}\lambda_0^{k-1}y_n + \frac{1}{n}\sum_{q=1}^{n-1}\lambda_0^{(k-1)(q-1)}u_q(y_n),$$ \hspace{1cm} (k = 1, \ldots, n). \hspace{1cm} (31)

**4. Bifurcation Analysis**

Now, we examine the effects of the $\epsilon$-perturbation on the bifurcation pattern for each $n$.

(i) The case where $n = 3$.

Let $z = re^{2\pi i \phi}$. Then the bifurcation function (30) can be reduced to

$$\lambda_1 \mu r e^{2\pi i \phi} - \frac{1}{2} \epsilon r e^{2\pi i \phi} + \bar{\lambda}_0 c_{02} r^2 e^{-2\pi i \phi} + g(r e^{2\pi i \phi}, r^{-2\pi i \phi}, \mu, \epsilon) = 0,$$ \hspace{1cm} (32)

where $\lambda_1 = 2\pi i$ and $g = O(|\mu|^2 r + |\mu| r^2 + r^3 + \epsilon r)$. If we separate the trivial solution $r = 0$, we are left with

$$\lambda_1 \mu - \frac{1}{2} \epsilon + \bar{\lambda}_0 c_{02} r e^{-6\pi i \phi} + \frac{e^{-2\pi i \phi}}{g(r e^{2\pi i \phi}, r^{-2\pi i \phi}, \mu, \epsilon)} = 0,$$ \hspace{1cm} (33)

Then (33) can be solved by the implicit function theorem near the origin. The solutions can be written as,

$$\mu = \frac{1}{2\pi} \left[ \left( \frac{\alpha_2}{2 \cdot 3^{1/2}} \right)^2 r^2 - \left( \frac{\epsilon}{2} \right)^{1/2} \right] + O(r^2),$$ \hspace{1cm} (34a)

$$\phi = \frac{-\beta}{3} + O(r), \text{ if } a_2 > 0,$$ \hspace{1cm} (34b)

where $\beta$ is determined from $\tan \beta = -4\pi \mu/\epsilon$. The stability of the three-cycle can be verified by the eigenvalues of the linear part of $(F^*)^3$,

$$\omega' = (F^*)^3(\omega) = (1 + 3\lambda_1 \mu)\omega - \frac{3}{2} \epsilon \omega + 3\lambda_0 c_{02} \bar{\omega} + O(|\omega|^3 + |\mu| |\omega|^2 + |\mu|^2 |\omega| + \epsilon |\omega|^2).$$ \hspace{1cm} (35)

If $\lambda_1$ and $\lambda_2$ are denoted as the eigenvalues of the Jacobian $A = \partial(\omega', \bar{\omega}')/\partial(\omega, \bar{\omega})$, it can be easily shown that $\lambda_1 + \lambda_2 = 0$ and $\lambda_1 - \lambda_2 = -\epsilon^2 - \pi^2 \mu^2 < 0$. Therefore the three-cycle is hyperbolic (saddle) and this branch is separated from the origin with the minimum distance of $3^{1/2}/\epsilon a^2$ (Fig. 1).
(ii) The case where \( n = 4 \).

In this case, the bifurcation function in the polar coordinate can be written as

\[
\lambda_1 \mu r e^{2 \pi i \phi} - \frac{1}{2} e r e^{2 \pi i \phi} + \lambda_0 b_{21} r^2 e^{2 \pi i \phi} + \lambda_0 b_{03} r^3 e^{-8 \pi i \phi} + O(r^4 + |\mu|^2 r^2 + |\mu|^2 + e^2 r) = 0, \tag{36}
\]

where \( b_{21}(0) = (a_2^2 + 3a_3)/8 \) and \( b_{03}(0) = (-a_2^2 + a_3)/8 \). After separating the trivial solution \( r = 0 \), (36) is reduced to

\[
\lambda_1 \mu - \frac{1}{2} e + \lambda_0 b_{21} r^2 + \lambda_0 b_{03} r^3 e^{-8 \pi i \phi} + O(r^4 + |\mu|^2 r^2 + |\mu|^2 + e^2) = 0 \tag{37}
\]

Equation (37) has nonzero solutions if \( \mu \) and \( \phi \) are of the order of \( r^2 \). By setting \( e = \varepsilon_0 r^2 + O(r^3) \) and \( \mu = \mu_0 r^2 + O(r^3) \), two pairs of solutions of (37) can be given in terms of \( \varepsilon_0 \) as

\[
\mu_0^{(1)} = \frac{b_{21}}{2 \pi} + \left[ \frac{b_{03}}{2 \pi} \right]^2 \left[ \frac{\varepsilon_0}{4 \pi} \right]^2 \frac{1}{2} \tag{38}
\]

\[
\phi_0^{(1)} = \begin{cases} \frac{-\beta}{4}, & \text{if } a_2 > 0, \\ \frac{1}{8} - \frac{\beta}{4}, & \text{if } a_2 < 0, \end{cases} \tag{39}
\]

where \( \beta \) is determined from \( \tan \beta = -\varepsilon_0/4\pi \left[ \frac{b_{03}}{2 \pi} \right]^2 - \left( \frac{\varepsilon_0}{4 \pi} \right)^2 \right)^{1/2} \). The stability analysis, similar to the case where \( n = 3 \) shows that if \( \mu_0^{(1)} \cdot \mu_0^{(2)} = \left[ a_2 (a_2^2 + a_3) + 2b_{03} \right] < 0 \), both branches are hyperbolic (Fig. 2). However, if \( \mu_0^{(1)} \cdot \mu_0^{(2)} > 0 \), then the branch with the larger \( r \) is stable and the other is hyperbolic (Fig. 3).

(iii) The case where \( n \geq 5 \).

Although this case can also be handled with the method similar to the above for each value of \( n \), we resort to a kind of trick of transforming the mapping equation into the polar form to obtain the general behavior of the solutions. Letting \( z = re^{2\pi i \phi} \) and \( F(z) = Re^{2\pi i \phi} \), we get from (12),

\[
Re^{2\pi i \phi} = \lambda_c(\mu) e^{2\pi i \phi} \left[ 1 + \sum_{m=1}^{P} \frac{b_{m+1,m}}{\lambda_c(\mu)} r^{2m} - \frac{b_{n-1,0}(\mu)}{\lambda_c(\mu)} e^{-2\pi in \phi} r^{n-2} \right] \tag{40}
\]
Letting $a = \sum_{m=1}^{p} \frac{b_{m+1,m} \mu^{2m}}{\lambda_{c}(\mu)}$ and $b = \frac{b_{m-1,0}(\mu)}{\lambda_{c}(\mu)} \times e^{-2\pi i \phi_{r}n-2}$ for notational convenience, (41) becomes

$$R = (1 - \epsilon)^{1/2} r |1 + a + b|$$

$$= (1 - \epsilon)^{1/2} r \left[1 + 2\Re a + 2\Re b + 2\Re (a \bar{b}) + |a|^{2} + |b|^{2}\right]^{1/2}$$

$$\simeq (1 - \epsilon)^{1/2} \left[1 + 2\Re b + 2\Re a + |a|^{2}\right]^{1/2}$$

where $\Re$ denotes the real part of the complex number and we kept only the terms of order up to that of $b$ in the last line.

But it can be shown that $2\Re a + |a|^{2} = 0$ from the fact that the normal form transformation should conserve the Jacobian value of the original mapping as follows. From (1) and its complex conjugate,

$$J = \frac{\partial(F(z), \bar{F}(z))}{\partial z}$$

$$= \left|\lambda_{c}(\mu) + \sum_{m=1}^{p} (m + 1)b_{m+1,m}(\mu)z^{m}\bar{z}^{m-1}\right|^{2}$$

$$\left|\sum_{m=1} b_{m+1,m}(\mu)z^{m}\bar{z}^{m-1} + (n - 1)b_{n-1,0}(\mu)z^{n-2}\right|^{2}$$

Setting $z = re^{2\pi i \phi}$,

$$J = 1 - \epsilon + 2\Re \left[\lambda_{c}(\mu) \sum_{m=1}^{p} (m + 1)b_{m+1,m}(\mu)r^{2m}\right] + \sum_{m=1}^{p} (m + 1)b_{m+1,m}(\mu)r^{2m}$$

$$\left|\sum_{m=1} b_{m+1,m}(\mu)r^{2m}\right|^{2}$$

where we kept only the terms of order up to that of $r^{n-2}$. But from the original mapping equation, $J = 1 - \epsilon$, which must be satisfied for all $r$. Thus

$$2\Re \left[\lambda_{c}(\mu) \sum_{m=1}^{p} (m + 1)b_{m+1,m}(\mu)r^{2m}\right] + \sum_{m=1}^{p} (m + 1)b_{m+1,m}(\mu)r^{2m}$$

$$\left|\sum_{m=1} b_{m+1,m}(\mu)r^{2m}\right|^{2} = 0$$

or

$$2 \sum_{m=1} (m + 1)\Re \left[\lambda_{c}(\mu)b_{m+1,m}(\mu)\right]r^{2m} + \sum_{m=2} \sum_{i+j=m} (m + 1)b_{i+1,i}(\mu)\bar{b}_{j+1,j}(\mu)r^{2m} = 0.$$
where $2m < n - 2$. Therefore we get

$$2\Re[\lambda_0(\mu) b_{m+1,m}(\mu)] + \sum_{i+j=m} b_{i+1,i}(\mu) \bar{b}_{j+1,j}(\mu) = 0. \quad (47)$$

In particular, when $n = 1$, $\Re[\lambda_0(\mu) b_{21}(\mu)] = 0$, which is in accordance with the direct calculation. From (47),

$$2\Re + |a|^2 = \sum_{m=1}^{2\Re b_{m+1,m}(\mu)} + \left| \sum_{m=1}^{2\Re b_{m+1,m}(\mu)} \right|^2 \left( 1 - \epsilon \right)^{-1} \sum_{m=1}^{2\Re b_{m+1,m}(\mu)} b_{m+1,m}(\mu) \lambda_0(\mu) b_{n-1,0}(\mu) e^{-2\pi im \phi r_{n-2}} \right. \left. + \sum_{i+j=m} b_{i+1,i}(\mu) b_{j+1,j}(\mu) \right) r^{2m} = 0. \quad (48)$$

Now (42) takes the simple form of

$$R = (1 - \epsilon)^{1/2} r (1 + 2\Re b)^{1/2} \approx (1 - \epsilon)^{1/2} r (1 + \Re b) \quad (49)$$

$$= r[(1 - \epsilon)^{1/2} + \Re \lambda_0(\mu) b_{n-1,0}(\mu) e^{-2\pi im \phi r_{n-2}}]]. \quad (50)$$

Since the bifurcation function $\lambda_0 = F*(z)$ is equivalent to $R = r$, $\Phi = \phi + \theta_0$, we get

$$R = r \left[ 1 - \frac{-\epsilon}{2} + \Re \lambda(\mu) b_{n-1,0}(\mu) e^{-2\pi im \phi r_{n-2}} \right] = r, \quad (51)$$

or

$$\epsilon = 2\Re \lambda(\mu) b_{n-1,0}(\mu) e^{-2\pi im \phi r_{n-2}}, \quad (52)$$

from which we deduce that $\epsilon = \mathcal{O}(r^{n-2})$ and two sets of values $\phi$ that correspond to the elliptic and hyperbolic fixed points in the numerical iteration of the mapping are possible.

The angular part of the mapping can be obtained as follows.

$$\Phi = \arg[\lambda_0 r e^{2\pi i \phi}]$$

$$+ \arg \left[ 1 + \sum_{m=1}^{2\Re b_{m+1,m}(\mu)} + \sum_{m=1}^{b_{n-1,0}(\mu) e^{-2\pi im \phi r_{n-2}}} \right] = \phi + \theta_0 + 2\pi \mu + \arg(1 + a + b)$$

$$= \phi + \theta_0 + 2\pi \mu + \tan^{-1} \frac{\Re(a+b)}{1 + \Re(a+b)} \quad (53)$$

Again keeping only the terms of order up to that of $b$,

$$\Phi = \phi + \theta_0 + 2\pi \mu + \sum_{k=0}^{\infty} \frac{(-1)^k}{2k + 1} \left( 3a[1 - \Re a + (\Re a)^2 + \cdots] + 3b \right)^{2k+1}$$

The bifurcation function $\Phi = \phi + \theta_0$ is

$$2\pi \mu + 3a + 3b \approx 0. \quad (55)$$

Since $a = \mathcal{O}(r^2)$ and $b = \mathcal{O}(r^{n-2})$, keeping only the lowest-order term we get

$$\epsilon = 2\Re \lambda_0 b_{n-1,0}(\mu) e^{-2\pi im \phi r_{n-2}}$$

$$2\pi \mu + 3\Re \lambda_0 b_{21}(\mu) r^2 = 0.$$

Thus we see that $\mu = \mathcal{O}(r^2)$ for $n \geq 5$.

5. Conclusions

The above analysis, which examines the effect of dissipative perturbation on the area-preserving mapping, shows that there is a close relationship between the unfolding pattern under the perturbation and the stability class of the $n$-cycle. The three-cycle is hyperbolic for $\epsilon = 0$ and structurally stable under the perturbation. The only difference is that, under the perturbation, the origin becomes separated from the three-cycle branch. Two situations arise when $n = 4$ in the area-preserving case. If $\mu^{(1)}_0 \cdot \mu^{(2)}_0 < 0$, then both branches are hyperbolic and are stable under the perturbation, similar to the case of $n = 3$. If $\mu^{(1)}_0 \cdot \mu^{(2)}_0 > 0$, however, Poincare–Birkhoff pairs of four-cycles are generated [Van der Weele et al., 1988]. These Poincare–Birkhoff pairs are stable under the perturbation except that the elliptic four-cycles in the area-preserving case become the stable ones. When $n \geq 5$, the Poincare–Birkhoff pairs of $n$-cycles are generated, which are retained in the dissipative case provided $\epsilon = \mathcal{O}(r^{n-2})$.

References


we can use (49) and (55) to get
\[ \frac{\epsilon}{2} = \Re[\lambda_0(b_{40}(0)e^{6\pi i\phi} + b_{13}(0)e^{-6\pi i\phi})r^3] \]
\[ = -\frac{3a_4}{8 \cdot 3^{1/2}} r^3 \cos 6\pi \phi \]
and
\[ 2\pi \mu = -3\left[\frac{b_{21}(0)}{\lambda_c(0)} r^2 + \frac{b_{40}(0)}{\lambda_c(0)} e^{6\pi i\phi} r^3 \right. \]
\[ \left. + \frac{b_{13}(0)}{\lambda_c(0)} e^{-6\pi i\phi} r^3 \right] \]
\[ = \frac{a_4}{2 \cdot 3^{1/2}} r^2 + \frac{5a_4}{8 \cdot 3^{1/2}} r^3 \sin 6\pi \phi. \]

The solution can be written as follows
\[ 2\pi \mu = \frac{a_4}{2 \cdot 3} r^2 + \frac{5}{3} \left( \frac{3a_4}{8 \cdot 3^{1/2}} \right) r^6 - \left( \frac{\epsilon}{2} \right)^{21/2} \]
\[ \tan 6\pi \phi = -\frac{5a_4}{3} r^2 / 3^{1/2} \]
\[ \frac{\epsilon}{2} \]
\[ \lambda_c(0) \]
\[ a = \frac{b_{21} (\mu)}{\lambda_c(\mu)} r^2, \]
\[ b = \frac{b_{40} (\mu)}{\lambda_c(\mu)} e^{6\pi i\phi} r^3 + \frac{b_{13} (\mu)}{\lambda_c(\mu)} e^{-6\pi i\phi} r^3, \]

**Appendix**

**n = 3 Case When a2 = 0**

Since the solution (34a) is not valid if \( a_2 = 0 \), we must consider the case separately. By letting
\[ b = \frac{b_{40} (\mu)}{\lambda_c(\mu)} e^{6\pi i\phi} r^3 + \frac{b_{13} (\mu)}{\lambda_c(\mu)} e^{-6\pi i\phi} r^3, \]
\[ a = \frac{b_{21} (\mu)}{\lambda_c(\mu)} r^2, \]

we can use (49) and (55) to get
\[ \frac{\epsilon}{2} = \Re[\lambda_0(b_{40}(0)e^{6\pi i\phi} + b_{13}(0)e^{-6\pi i\phi})r^3] \]
\[ = -\frac{3a_4}{8 \cdot 3^{1/2}} r^3 \cos 6\pi \phi \]
and
\[ 2\pi \mu = -3\left[\frac{b_{21}(0)}{\lambda_c(0)} r^2 + \frac{b_{40}(0)}{\lambda_c(0)} e^{6\pi i\phi} r^3 \right. \]
\[ \left. + \frac{b_{13}(0)}{\lambda_c(0)} e^{-6\pi i\phi} r^3 \right] \]
\[ = \frac{a_4}{2 \cdot 3^{1/2}} r^2 + \frac{5a_4}{8 \cdot 3^{1/2}} r^3 \sin 6\pi \phi. \]

The solution can be written as follows
\[ 2\pi \mu = \frac{a_4}{2 \cdot 3} r^2 + \frac{5}{3} \left( \frac{3a_4}{8 \cdot 3^{1/2}} \right) r^6 - \left( \frac{\epsilon}{2} \right)^{21/2} \]
\[ \tan 6\pi \phi = -\frac{5a_4}{3} r^2 / 3^{1/2} \]
\[ \frac{\epsilon}{2} \]
\[ \lambda_c(0) \]
\[ a = \frac{b_{21} (\mu)}{\lambda_c(\mu)} r^2, \]
\[ b = \frac{b_{40} (\mu)}{\lambda_c(\mu)} e^{6\pi i\phi} r^3 + \frac{b_{13} (\mu)}{\lambda_c(\mu)} e^{-6\pi i\phi} r^3, \]