Scaling behavior at the onset of chaos in the logistic map driven by colored noise

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Abstract
The effects of additive colored noise near the onset of chaos have been studied for one-dimensional logistic maps. The scaling form of the Lyapunov exponent $\lambda$ near the transition point was determined in terms of the noise amplitude $\sigma$ and the correlation time $\tau$, and the critical exponent $\alpha$ associated with $\tau$ was evaluated. Analysis shows that a certain approximation, which we call the effective white noise approximation, followed by the renormalization group treatment, leads to essentially the same structure of the scaling function as that determined by the numerical method.

The effects of noise on the dynamical behavior of nonlinear systems have been studied by several authors [1–3]. One well known study involves an analytical and numerical investigation of the effects of white noise on the behavior of the Lyapunov exponent $\lambda$ at the onset of transition to chaos in one-dimensional logistic maps which exhibit a period-doubling cascade [1,4]. The characteristic features observed in this system are: (i) noise enhances the chaotic behavior, (ii) the detailed structure of the Lyapunov exponent $\lambda$ disappears, while the global structure is relatively unchanged, and (iii) the slope of $\lambda$ at the transition point $r_c$ decreases as the noise amplitude increases. Furthermore, $\lambda$ satisfies a universal scaling function in terms of the noise amplitude $\sigma$ and control parameter $\tilde{r} = r - r_c$.

$$\lambda = \sigma^\alpha L(\tilde{r}\sigma^{-\gamma}). \quad (1)$$

This scaling form is confirmed by the renormalization group treatment [4,5] which is similar to that applied to the one-dimensional Ising model system.

The autocorrelation of white noise is, by its definition, a $\delta$-function, $\langle \xi_n \xi_{n'} \rangle = \sigma^2 \delta_{n,n'}$, where $\sigma$ is the amplitude of the noise. So white noise is a representation of a highly simplified form of external fluctuations. However, at a microscopic level, all external fluctuations have finite correlation times in general. Recent studies show that if the correlation time is short, the effects of colored noise on the dynamical behavior are essentially the same as those of white noise [6]. However, when the correlation is comparable to, or longer than, the characteristic time scale of the system, its effects can be qualitatively different from those of white noise, which has not been clearly understood yet.

From the point of view of the fluctuation-dissipation theorem, white noise is a first order approximated form of external fluctuation, and colored

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noise whose autocorrelation is of exponential type is a second order approximation [7]. Besides, the importance of understanding the role of colored noise has grown, specially in the fields of laser theory [8] and nonequilibrium phase transitions in liquid crystals [9], etc.

In this paper we use colored noise and study the effects of colored noise on the dynamics of a logistic map both numerically and analytically. Special focus is on the study of the scaling behavior of the Lyapunov exponent $\lambda$ which is still an invariant for the system with external noise.

The main result to be reported here is that not only the noise amplitude $\sigma$, but also the correlation time $\tau$ behaves as a scaling variable in the case of colored noise if the correlation time is sufficiently long. Quoting the result first, the dependence of the Lyapunov exponent $\lambda$ on $\tau$ is negligible if $\tau$ is short ($\tau < 1$) and the scaling form of $\lambda$ is the same as Eq. (1), that for white noise, i.e., $\lambda$ depends only on $\sigma$ and the control parameter $\tilde{r}$. But for sufficiently long $\tau$, the scaling form of $\lambda$ becomes asymptotically

$$\lambda(\tilde{r}; \sigma, \tau) = \left(\frac{\sigma}{\tau'}\right)^{\theta L[\tilde{r}(\sigma/\tau')^{-\gamma}]}.$$

where $\theta$ and $\gamma$ are the same exponents as defined in the white noise case whose numerical values are known to be $\theta = 0.37$ and $\gamma = 0.82$ respectively, and $\alpha$ is a new critical exponent associated with the correlation time $\tau$. Our approximate theory shows that $\alpha$ is $1/2$ exactly in the limit of large $\tau$. This $\tau$ dependent scaling behavior is supported by numerical simulations. Notice that the above scaling form is similar to that derived for the case of white noise except that $\sigma$ is replaced by $\sigma/\tau'$. Another change observed for the behavior of $\lambda$ in the case of colored noise is that the fine structure of $\lambda$ which was suppressed by white noise reappears as the correlation time $\tau$ increases with a fixed, finite amplitude $\sigma$.

A rigorous derivation of the scaling function of $\lambda$ is unavailable, but by applying a certain approximation analogous to the mean field approximation which has been widely used in the theory of critical phenomena, we were able to obtain the correct structure of the scaling function for the whole range of $\tau$, as was confirmed from the numerical study.

We start with the well known one-dimensional logistic map with additive colored noise,

$$x_{n+1} = r x_n (1 - x_n) + \epsilon_n,$$

where $\{\epsilon_n\}$ is a sequence of colored noise. The discrete colored noise can be generated using a first order autoregressive process, which is a discrete version of the Ornstein–Uhlenbeck process. The first order autoregressive process $\{\epsilon_n\}$ is defined as follows. Let $\xi_0, \xi_1, \xi_2, \ldots$ be a sequence of uncorrelated random variables with $\langle \xi_n \rangle = 0, n \geq 0$, and

$$\langle \xi_n^2 \rangle = \sigma^2 / (1 - \rho^2), \quad n = 0,$$

$$= \sigma^2, \quad n \geq 1,$$

where $\rho^2 < 1$, and define

$$\epsilon_0 = \xi_0, \quad \epsilon_n = \rho \epsilon_{n-1} + \xi_n, \quad n \geq 1.$$

Then the process $\{\epsilon_n, n \geq 0\}$ is called a first order autoregressive process and satisfies $\langle \epsilon_n \rangle = 0$ and

$$\langle \epsilon_n \epsilon_{n'} \rangle = \frac{\sigma^2}{1 - \rho^2} \rho^{|n-n'|} \equiv \sigma^2 \rho^{|n-n'|},$$

where $\rho'$ defined by $\sigma^2 / (1 - \rho^2)^{1/2}$ is the true intensity of the colored noise and $\rho$ is related to the correlation time $\tau$ by $\rho = e^{-1/\tau}$. Notice that if we want to obtain a sequence of colored noise $\{\epsilon_n\}$ with intensity $\sigma'$ and correlation time $\tau$ we have to use a sequence of white noise $\{\xi_n\}$ with intensity $\sigma = \sigma' (1 - \rho^2)^{1/2}$. In the limit of $\tau \to 0$, the process $\epsilon_n$ reduces to white noise,

$$\lim_{\tau \to 0} \langle \epsilon_n \epsilon_{n'} \rangle = \sigma^2 \delta_{nn'}.$$

The scaling behavior is known to be independent of the form of the distribution of noise [2]. Here, however, the $\{\epsilon_n\}$ are assumed to have a Gaussian probability distribution for later analytical study. Since a Gaussian process is transformed into another Gaussian through a linear transformation, if $\{\xi_n\}$ have a Gaussian distribution, then $\{\epsilon_n\}$ also have a Gaussian form.

Next, we shall study how the self-similarity of the dynamics as a function of the control parameter allows for a scaling behavior of the Lyapunov exponent $\lambda$ in the case of colored noise, by examining $\lambda$ which is defined by [5]

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \log \left\langle \frac{d}{dx_0} \left(f^n(x_0; r)\right)_\rho \right\rangle_{x_0},$$

where $f$, in our case, is the logistic map defined in Eq. (3), $f^n$ is the $n$th iterate of $f$, and $\langle \cdot \rangle_\rho$ and $\langle \cdot \rangle_{x_0}$
Fig. 1. (a) The scaling behavior of $\lambda$ is shown to be identical to the case of white noise, Eq. (1), for small $\tau$. $\lambda(\sigma^2, \tau)$ is plotted against $\tilde{F}(\sigma^2) - \tau$ for six different pairs of $(\sigma^2, \tau)$: $(\times) (10^{-6}, 0.01)$, $(\times) (10^{-6}, 1.0)$, $(\times) (10^{-8}, 0.01)$, $(\times) (10^{-8}, 1.0)$, $(\times) (10^{-10}, 0.01)$, $(\times) (10^{-10}, 1.0)$. (b) Numerically determined scaling function for large $\tau$ $(\tau \geq 10)$. Eq. (2). $\lambda(\sigma^2/\sigma^0 - \tau)$ is plotted against $F(\sigma^2/\sigma^0) - \tau$ with $a = 0.7$ for six different pairs of $(\sigma^2, \tau)$: $(\times) (10^{-6}, 10.0)$, $(\times) (10^{-6}, 1000.0)$, $(\times) (10^{-8}, 10.0)$, $(\times) (10^{-8}, 1000.0)$, $(\times) (10^{-10}, 10.0)$, $(\times) (10^{-10}, 1000.0)$.

denote the averages taken with respect to the noise distribution and the initial condition, respectively. In particular, we evaluate the scaling function of $\lambda$ near the point of the transition to chaos both by an analytical and a numerical method. To determine $\lambda$ from an observed time series numerically, we can apply either the method developed by Bryant, Brown and Abarbanel [12], or the concept of invariant distribution established by Rochester and White [13]. The higher order Taylor series expansion technique of Bryant et al. which can construct a nonlinear form of local mappings rather than a linear one from the observed time series, followed by the determination of the local Jacobian, is suitable for higher dimensional systems which may have more than one positive exponents. But we find that the invariant distribution is more useful and effective for the case of a one-dimensional system with small random fluctuations. Since there exists a unique invariant distribution which characterizes attractors of a one-dimensional map even in the presence of noise [13], and dynamical averaging, performed numerically, and statistical averaging, performed with the invariant distribution are identical, we choose the dynamical averaging method for the evaluation of $\lambda$, which is often used in statistical mechanics to define a probability distribution. The number of trajectories $N$ used for dynamical averaging is $10^7$ for each $(\sigma^2, \tau)$.

The transition point to chaos enhances as the noise amplitude increases and this enhancement satisfies a characteristic relation in terms of the noise amplitude. But the enhancement is reduced as the correlation time of colored noise $\tau$ increases at a fixed value of the noise amplitude. Moreover, the slope of $\lambda$ at the transition point increases as $\tau$ increases. The numerically determined scaling function is shown in Fig. 1 for various values of $\sigma^2$ and $\tau$. Fig. 1a shows the scaling behavior of $\lambda$ for small $\tau$ $(\tau < 1)$. In this region, $\lambda$ is insensitive to $\tau$ and the form of the scaling function is essentially the same as that used in the case of white noise, Eq. (1). This is a consequence of the fact that, in the scaling region, after sufficiently many decimation steps in the renormalization group (RG) procedure, finite time scales to arbitrarily close to zero, and the scaling equation for the white noise is recovered even if at the outset the noise is colored.

On the other hand, for sufficiently large $\tau$ $(\tau \geq 10)$, as can be seen in Fig. 1b, $\lambda$ has a certain dependence on $\tau$, which can be expressed as Eq. (2). In the latter
case $\alpha$ is determined by adjusting the slope near the transition point so that all the scaling functions with different $\tau$ collapse to a single smooth curve using the same numerical values of $\theta$ and $\gamma$ as Crutchfield et al. [1] and Schraiman et al. [4]. $\alpha$ was found to be 0.7 following this procedure. However, because the variation of the slope is quite insensitive to the variation of $\alpha$ it is difficult to determine the optimum value of $\alpha$ unequivocally. From these numerical studies we found that the effects of colored noise with sufficiently large correlation time $\tau$ are: (i) to diminish the enhancement of the onset of chaos, (ii) to reduce the scaling region, (iii) to increase the slope of $\lambda$, and, furthermore, to make the fine structure of $\lambda$ reappear both in the periodic and chaotic regimes as can be seen in Fig. 2. This figure shows that the structure of $\lambda$ becomes closer to that of the deterministic map as $\tau$ increases.

To understand intuitively the effect of colored noise on the cascade bifurcation, we consider the power spectrum of colored noise which is superposed on that of the logistic map at the transition point. As was already pointed out by Crutchfield et al. [2], in the case of white noise, all the peaks of subharmonics whose magnitudes are smaller than the noise level $\sigma$ are suppressed uniformly. However, in the case of exponentially correlated colored noise, the power spectrum of the noise is a Lorentzian whose bandwidth is approximately $1/\tau$, and the spectral lines corresponding to higher subharmonics become less suppressed as $\tau$ increases. Hence higher order bifurcations reappear as is shown in Fig. 2.

A rigorous RG treatment for colored noise is not available at present. However, an approximation scheme, which we call the effective white noise approximation, made it possible to apply the RG treatment which is identical to that developed by Schraiman and coworkers. From this RG treatment, the scaling form of $\lambda$, which depends not only on $\sigma$ and $r$, but also on $\tau$, in the vicinity of the transition point to chaos was derived, and this scaling form was confirmed by numerical simulations.

The analysis starts by specifying the probability distribution function of a sequence of Gaussian colored noise, $P(\epsilon)$. If the process is temporally homogeneous (stationary), the probability distribution is given by

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**Fig. 2.** Plot of $\lambda$ against the control parameter $r$ for two values of $\tau$, 0.1 and 50.0 at $\sigma = 1.0 \times 10^{-6}$. As $\tau$ increases more bifurcations are observed.

**Fig. 3.** An example of the power spectrum of colored noise with $\tau = 5.0$ and that of the corresponding effective white noise, Eq. (11). The units of the $y$-axis are arbitrary. The amplitude of the effective white noise is reduced by a factor of $[(1 - \rho^2)/(1 + \rho^2)]^{1/2}$. 

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Fig. 4. The scaling function obtained from the effective white noise approximation. (a) Numerical plots of $\lambda |\sigma' / g(\tau)|^{-\beta}$ against $\tilde{r}(\sigma' / g(\tau))^{-1}$ at fixed $\sigma' = 1.0 \times 10^{-8}$ for seven different values of $\tau$: (o) 0.01, (+) 0.1, (□) 1.0, (△) 10.0, (▲) 100.0, (○) 1000.0. (b) Numerical plots of $\lambda |\sigma' / g(\tau)|^{-\beta}$ against $\tilde{r}(\sigma' / g(\tau))^{-1}$ at fixed $\tau = 10.0$ for three levels of the noise amplitude $\sigma'$: (o) $1.0 \times 10^{-6}$, (+) $1.0 \times 10^{-8}$, (□) $1.0 \times 10^{-10}$.

$$P\{\epsilon_j\} = \frac{1}{(2\pi\sigma'^2)^{n/2}} \prod_{j=1}^{n-1} \left(1 - \rho^2\right)^{1/2} \times \exp\left(-\frac{1}{2\sigma'^2} \epsilon_1^2 - \frac{1}{2\sigma'^2} \sum_{j=1}^{n-1} \frac{(\epsilon_{j-1} - \rho \epsilon_j)^2}{1 - \rho^2}\right).$$

(9)

where $\sigma'$ is the intensity of colored noise. It turns out that the probability distribution is identical to the $n$-variate Gaussian probability distribution of the Ornstein–Uhlenbeck process, which describes the velocity of a Brownian particle, with constant time step. Expanding the arguments in the exponential of Eq. (9), we obtain

$$-\frac{(1 + \rho^2)}{2\sigma'^2(1 - \rho^2)} \sum_{j=1}^{n} \epsilon_j^2$$

$$+ \frac{1}{2\sigma'^2(1 - \rho^2)} \left(\rho^2 \epsilon_1^2 - \rho^2 \epsilon_n^2 + 2\rho \sum_{j=1}^{n-1} \epsilon_j \epsilon_{j+1}\right).$$

(10)

It would be a reasonable approximation to use $\sum_{j=1}^{n} \epsilon_j \epsilon_{j+1} \approx (n - 1) \epsilon_1 \epsilon_{n-1}$ for $n$ sufficiently large. To make the process a mutually independent homogeneous one, we further approximate $\epsilon_1^2$ and $\epsilon_n^2$ in the second term of Eq. (10) as $\sigma'^2$.

Finally, normalizing this approximated form of distribution, we obtain the following effective Gaussian probability distribution $P\{\epsilon_j\}$,

$$P\{\epsilon_j\} \approx \frac{1}{(2\pi\eta^2)^{n/2}} \exp\left(-\frac{1}{2\eta^2} \sum_{j=1}^{n} \epsilon_j^2\right).$$

(11)

where

$$\eta^2 = \sigma'^2 \frac{1 - \rho^2}{1 + \rho^2}. $$

So this approximation procedure reduces colored noise with amplitude $\sigma'$ and correlation time $\tau$ to an effective white noise with an amplitude $\eta$ which depends on $\tau$. This approximation can be considered to be similar to the well known mean field approximation. Fig. 3 shows the power spectra of the original colored noise and the effective white noise whose amplitude is reduced by a factor of $(1 - \rho^2)/(1 + \rho^2)$. As can be seen from this figure, the effective white noise approximation suppresses higher order subharmonics more than it should do. However, the discrepancies between the effects of the original colored noise and those of
the corresponding effective white noise are negligibly small for small $\tau$, and are not significantly large even for large $\tau$.

This effective Gaussian probability distribution $P\{e_i\}$ enables us to apply the RG method developed by Schraiman, Wayne and Martin [4,5]. Just showing the result, the scaling form of the Lyapunov exponent $\lambda$ becomes

$$\lambda(\bar{r}; \sigma', \tau) = (\sigma'/g(\tau))^\alpha L[\bar{r}(\sigma'/g(\tau))^{-\gamma}].$$

(12)

where $g(\tau) = (1 + \rho^2)/(1 - \rho^2) = \coth(1/\tau)$. The scaling function shows qualitatively correct limiting behavior as $\tau \to \infty$ and $\tau \to 0$. As $\tau \to 0$, $g(\tau) \to 1$ and the scaling function recovers the form of that for the case of white noise, i.e., Eq. (1), and as $\tau \to \infty$ with a fixed, finite value of $\sigma'$, $g(\tau)$ is reduced to $\tau^{1/2}$ and the slope of $\lambda$ goes to infinity at the transition point as is observed in the case of the deterministic system. The scaling form in the limit of large $\tau$ becomes

$$\lambda(\bar{r}; \sigma', \tau) = (\sigma'/\tau^{1/2})^\alpha L[\bar{r}(\sigma'/\tau^{1/2})^{-\gamma}].$$

(13)

So in the above approximation, the critical exponent $\alpha$ for large $\tau$ is 0.5.

We plot $\lambda(\sigma'/g(\tau))^{-\gamma}$ against $\bar{r}(\sigma'/g(\tau))^{-\gamma}$ for seven different values of $\tau$ at $\sigma' = 1.0 \times 10^{-8}$ in Fig. 4a. As can be seen in this diagram, the effective white noise approximation, from which $\alpha = 1/2$ is obtained in the limit of large $\tau$, does not deteriorate the scaling behavior of $\lambda$. Considering the fact that the slope of $\lambda$ at the transition point is not sensitive to the variation of $\alpha$ in the case of large $\tau$, our approximate theory essentially shows the correct scaling behavior of $\lambda$ in the vicinity of the transition point for the whole range of $\tau$.

Finally, we show almost the same plot as Fig. 4a, but in this case, for three different levels of noise amplitude $\sigma'$ with fixed $\tau = 10.0$ in Fig. 4b. Notice that most of the data points lie on a single smooth curve in the scaling regime, which guarantees that the effective white noise approximation leads to, at least, qualitatively correct scaling behavior.

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References