NEW CHARACTERISTIC RELATIONS 
IN TYPE-II AND III INTERMITTENCY

M. O. KIM and HOYUN LEE
Department of Physics, Chungnam National University, Taejon 305-764, Korea

CHIL-MIN KIM
Department of Physics, Pai Chai University, Taejon, 302-735, Korea

HYUN-SOO PANG and EOK-KYUN LEE
Department of Chemistry, Korea Advanced Institute of Science and Technology, Taejon, 305-701, Korea

O. J. KWON
Department of Science Education, Kongju National University of Education, Kongju, Chungnam, 314-060, Korea

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We obtained new characteristic relations in Type-II and III intermittencies according to the reinjection probability distribution. When the reinjection probability distribution is fixed at the lower bound of reinjection, the critical exponents are $-1$, as is well known. However when the reinjection probability distribution is uniform, the critical exponent is $-1/2$, and when it is of form $1/\sqrt{\varepsilon}$, $-3/4$. On the other hand, if the square root of $\Delta$, which represents the lower bound of reinjection, is much smaller than the control parameter $\varepsilon$, i.e. $\varepsilon \gg \Delta^{1/2}$, critical exponent is always $-1$, independent of the reinjection probability distribution. Those critical exponents are confirmed by numerical simulation study.

1. Introduction

The intermittent transitions from chaos to stable orbit (or vice versa) were discussed by Pomeau and Manneville in connection with the Lorenz model and circle map. They classified those into three types, Type-I, II, and III according to the local Poincaré map and pointed out that they arise when tangent bifurcation occurs [Pomeau & Manneville, 1980]. When a parameter decreases slightly from the tangent bifurcation point, the stable periodic orbits lose their stability and intermittently appearing chaotic bursts can be observed between quasiperiodic (laminar) signals. When the parameter decreases more, chaotic bursts appear more frequently. And the signal develops into chaos when the parameter fully decreases. Intermittencies are characterized by their intermittent changes between chaotic signals and quasiregular signals, and they have their own characteristic relation depending on their type; for example, the characteristic relation of Type-I intermittency is $\langle l \rangle \propto \varepsilon^{-1/2}$, and that of Type-II and Type-III is $\langle l \rangle \propto \varepsilon^{-1}$, where $\varepsilon$ is the control parameter at critical point where tangent bifurcation occurs, and $\langle l \rangle$ is an average laminar length. The relations are obtained by considering the local Poincaré map, assuming that the reinjection probability distribution (RPD) is uniform and $\Delta \ll \varepsilon^{-1/2}$ where $\Delta$ is the lower bound of the reinjection (LBR) [Hirsch et al., 1982].
Recently, however, it was reported that the RPDs are not always uniform and so Type-I intermittency can have various characteristic relations such as $-\ln \epsilon$ and $\epsilon^{-\nu}$, where $\nu$ can vary from 1/4 to 1/2 [Kim et al., 1994]. This reveals that the RPD acts as one of the important factors in determining critical exponents and that it is expected to have a similar effect on scaling behavior of the other intermittencies. Here we report various new characteristic relations in Type-II and III intermittency according to various forms of the RPD.

2. Analytical Solutions

We first discuss the result of Pomeau and Manneville on the characteristic relation of Type-II and III intermittency of which local Poincaré maps are given by $x_{n+1} = (1 + \epsilon)x_n + ax^3_n$ and $x_{n+1} = -(1 + \epsilon)x_n - ax^3_n$ respectively, where $\epsilon > 0$ and $a > 0$. We assume that the flow direction is positive without loss of generality. In the local Poincaré maps, the intermittencies appear when $\epsilon$ is slightly larger than 0. In the present work, we discuss Type-II intermittency only. The local Poincaré map of Type-III intermittency can be described by $x_{n+2} = (1 + 2\epsilon)x_n + bx^3_n$ which has the same structure as that of Type-II intermittency [Schuster, 1987].

We first denote the LBR as $x_L$ and assume that $x_L > x_c$, where $x_c$ is the tangent point. Then the average laminar length for Type-II intermittency depends on $\epsilon^{-1}\ln \Delta$, where $\Delta = x_L - x_c$. When $\epsilon \gg \Delta^{1/2}$, the average laminar length is given by $\langle l \rangle \propto \epsilon^{-1}$, if the RPD is assumed to be uniform between $x_L$ and the gate $c$.

We now discuss the characteristic relations of Type-II intermittency according to the RPD. If $x_{in}$ is a re-injection point in the return map in the laminar region, the laminar length $l(x_{in}, c)$ which depends on $x_{in}$ is obtained in the long laminar length approximation as,

$$
l(x_{in}, c) = \frac{2 \ln \left[ \frac{c}{x_{in}} \right] - \ln \left[ \frac{(ac^2 + \epsilon)}{(ax^2_{in} + \epsilon)} \right]}{2\epsilon}$$

Then the average laminar length $\langle l \rangle$ for a given RPD $P(x_{in})$ is given by

$$\langle l \rangle = \int_{\Delta}^{\infty} l(x_{in}, c)P(x_{in})dx_{in}$$

We here normalize $P(x_{in})$ to 1 and assume that the LBR is above the tangent point.

We restrict our discussion to three Types of the RPD which are integrable: Uniform, fixed at LBR, and $1/\sqrt{x_{in}}$ which are monotonically decreasing functions of $x_{in}$. If the RPD is uniform, the normalized RPD can be expressed in $P(x_{in}) = 1/(c - \Delta)$. If re-injection is fixed at LBR, $P(x_{in}) = \delta(x_{in} - \Delta)$, and finally if it is of the form $1/\sqrt{x_{in}}$, the normalized RPD is $P(x_{in}) = 1/(2\sqrt{c - \Delta \sqrt{x_{in} - \Delta}})$.

We first consider the case of uniform RPD. We can obtain the following characteristic relation from Eq. (2).

\[
\langle l \rangle^u = \tan^{-1} \left[ \frac{c}{\sqrt{\epsilon}} \right] - \tan^{-1} \left[ \frac{\Delta}{\sqrt{\epsilon}} \right] \sqrt{\epsilon(a - \Delta)} \\
\frac{2\Delta \ln \left[ \frac{c}{\Delta} \right] - \Delta \ln \left[ \frac{(ac^2 + \epsilon)}{(a\Delta^2 + \epsilon)} \right]}{2\epsilon(a - \Delta)}
\]

Here we assume that the LBR is fixed without loss of generality. Then the second term becomes negligible because the numerator of the second term goes to zero in the limit $\epsilon \to 0$. Then the average laminar length has such characteristic relation as $\langle l \rangle \propto \epsilon^{-1/2}$. This Type of critical exponent appears in Type-I intermittency when the LBR is below the tangent point or the RPD is fixed at the tangent point. But this is a new result in Type-II intermittency. On the contrary, if we just assume $\epsilon \gg \Delta^{1/2}$, the second term is not negligible. In this case, the dominant term is the second and the characteristic relation is of the form $\langle l \rangle \propto \epsilon^{-1}$ which is the very result of Pomeau and Manneville.

We next consider the case that the RPD is fixed at the LBR. The characteristic relation is

\[
\langle l \rangle^f = \frac{2 \ln \left[ \frac{c}{\Delta} \right] - \ln \left[ \frac{(ac^2 + \epsilon)}{(a\Delta^2 + \epsilon)} \right]}{2\epsilon}
\]

The above characteristic relation always reduces to $\langle l \rangle \propto \epsilon^{-1}$ regardless of whether $\epsilon \gg \Delta^{1/2}$ or not. This critical exponent is the same as that of Pomeau and Manneville.

We further consider the nonuniform RPD of the form $1/(2\sqrt{c - \Delta \sqrt{x_{in} - \Delta}})$, for which the analytic calculation of characteristic relation is
possible. The characteristic relation is

\[
(l)_{n}^{u} = \frac{2}{\epsilon} \sqrt{\frac{\Delta}{\Delta - c}} \tan^{-1} \left( \frac{c - \Delta}{\sqrt{\Delta(\Delta - c)}} \right) + \frac{(-1)^{7/4}}{\epsilon} \sqrt{\frac{\sqrt{\epsilon - I\Delta \sqrt{\alpha}}}{\sqrt{\alpha(\epsilon - \Delta)}}} \tan^{-1} \left( \frac{I\sqrt{\alpha(c - \Delta)}}{\sqrt{\epsilon - I\Delta \sqrt{\alpha}}} \right) \\
+ \frac{(-1)^{3/4}(\Delta \sqrt{\alpha} - I\sqrt{\epsilon})}{\epsilon \sqrt{\alpha(c - \Delta)(\sqrt{\epsilon + I\Delta \sqrt{\alpha}})}} \tan^{-1} \left( \frac{I^{1/2} \sqrt{\alpha(c - \Delta)}}{\sqrt{\epsilon + I\Delta \sqrt{\alpha}}} \right)
\]

(5)

This characteristic relation has the form of \((l)_{n}^{u} \propto \epsilon^{-3/4}\) in the limit \(\epsilon \to 0\) when \(\Delta\) is fixed, because the exponent of \(\epsilon\) of the first term is dominant due to the same reason appeared in case of the uniform RPD. It is another new Type of critical exponent. However, if \(\epsilon \gg \Delta^{1/2}\), which is the condition imposed by Pomeau and Manneville, the characteristic relation is of the form \((l) \propto \epsilon^{-1}\) as discussed above.

We are now able to give some general arguments on the characteristic relations of Type-II intermittency in the limit of \(\epsilon \to 0\). The intermittency has various critical exponents according to the RPD. If the RPD is uniform, it is \(-1/2\), if it is fixed at LBR, it is \(-1\), and finally if it is of the form \(1/\sqrt{x_{\text{in}}}\), it becomes \(-3/4\). However if we impose the condition \(\epsilon \gg \Delta^{1/2}\), the characteristic relation is always \((l) \propto \epsilon^{-1}\) as was observed by Pomeau and Manneville.

3. Numerical Results

Now we study an illustrating model map to confirm the above results, given by

\[
x_{n+1} = (1 + \epsilon)x_{n} + dx_{n}^{3}(1 - x_{n}^{2}).
\]

(6)

To give rise to sufficiently small \(\Delta\), we set \(d_{c} = 1.902110391216677\). On this condition, when \(\epsilon = 0\), the lower bound of the reinjection \(\Delta\) is approximately \(10^{-10}\), and the slope of local Poincaré map at the tangent point is 1. The bifurcation diagram according to \(\epsilon\) varies from -0.35 to 0.05 as is shown in Fig. 1. In this figure, Type-II intermittency occurs near period-1 window. The window occurs when \(\epsilon\) is negative, and the tangent bifurcation occurs at \(\epsilon = 0\). When the parameter \(\epsilon\) increases slightly, intermittent chaotic bursts appear between quasiregular signals. In this region, the chaotic band is bounded in the range \([x_{L}, x_{U}]\), and so the band width is \(x_{U} - x_{L}\). Here \(x_{U} = F(\epsilon, d; x_{M})\) is the upper bound of the chaotic band, where \(x_{M}\) is a point satisfying \(dF(\epsilon, d; x_{M})/dx = 0\), and \(x_{L} = F(\epsilon, d; x_{U})\) [see Fig. 2]. In the model map, as \(\epsilon\) varies, the LBR changes slightly. This is due to the change of the upper bound which depends on \(d\) as well as on \(\epsilon\). So to maintain the LBR to be always at a value of \(1.0 \times 10^{-10}\), which is fixed up to 16 significant figures, the parameter \(d\) has to be changed slightly. The value of \(d\) is obtained numerically using the bisector method. If we let the tangent point where the local Poincaré map meets the diagonal

![Bifurcation diagram](image)

Fig. 1  Bifurcation diagram of the model map, Eq. (6), in the range of \(-0.35 < \epsilon < 0.05\) at \(d = 1.902110391216677\).
line be $x_c$, the LBR is $\Delta = x_L - x_c$. Figure 2 shows the local Poincaré map and the LBR when Type-II intermittency occurs at $\epsilon = 10^{-5}$, $d = 1.85207870$, and $\Delta = 0.1$. The map shows the LBR, the upper bound of the chaotic band, the tangent point, and the transition from fixed point to chaos through Type-II intermittency.

The RPD of the model map is of the Type $1/\sqrt{x}$, as is confirmed from a histogram of re-injection probability as shown in Fig. 3. In the figure, each bar represents the number of re-injections which enter each region from $x_i$ to $x_{i+1}$. By counting the re-injection number for each region $[x_i, x_{i+1}]$, we obtain the normalized RPD as $P(x_n) = 1/(2\sqrt{c - \Delta \sqrt{x - \Delta}})$. The region of the histogram from the LBR to the gate ($\Delta \leq x \leq \Delta + 0.01$) is divided into 100 sectors and the total number of re-injection points is $10^6$.

To get an analytic solution from the map, the local Poincaré map near the tangent point is obtained using the Taylor expansion. The local Poincaré map is given by $x_{n+1} = (1 + \epsilon)x_n + ax_n^3$, where $a = d_c + (d - d_c)/4$ and $d = d(\epsilon)$. Using the map, the analytical form of the average laminar
length according to $\epsilon$ is obtained, which appears as the solid line II in Fig. 4, when RPD is of the form $1/\sqrt{\epsilon}$. The result obtained from Eq. (6) agrees well with the numerical experiments appearing in dots on Line II. The critical exponent is $-3/4$ in the limit of $\epsilon \to 0$ for a finite value of $\Delta$, and it is $-1$ when $\epsilon \gg \Delta^{1/2}$, which agrees well with the ones obtained analytically.

Next, to obtain the average laminar length numerically in the case of uniform RPD, we generate the re-injection points randomly by using the computer's random number generator, and to obtain that of fixed RPD, we fix the re-injections at the LBR. From the map, we obtain the average laminar lengths in the cases of uniform and fixed RPDs according to $\epsilon$ as are shown by the dots on Line III and Line I, respectively, in Fig. 4. The results agree well with the analytic calculations of the two lines obtained from Eqs. (3) and (4). As is shown in the figures, the slopes of uniform and fixed RPD are $-1/2$ and $-1$ respectively, in the limit of $\epsilon \to 0$. And the slopes are always $-1$ when $\epsilon \gg \Delta^{1/2}$, as in the result of Pomeau and Manneville. Here the two exponents $-3/4$ and $-1/2$ are new critical ones which appear in the limit $\epsilon \ll \Delta^{1/2}$.

We further obtain the characteristic relations of Type-III intermittency from the map $x_{n+1} = -(1 + \epsilon)x_n - dx_n^3(1 - x_n^2)$. The map can be converted into $x_{n+2} = F(F(x_n))$, so that the local Poincaré map can be written as $x_{n+2} = (1 + 2\epsilon)x_n + ax_n^3$, which is the same as that of Type-II intermittency. In the numerical experiment, we obtain the laminar lengths using a converted map and the results are compared with analytic calculations obtained from Eqs. (3)–(5) for the three cases of the RPD as are discussed in Type-II intermittency. All the critical exponents obtained in the converted map are the same as those of Type-II intermittency, if the LBR is above the tangent point and the value of the LBR is $10^{-10}$. The numerically obtained average laminar lengths according to $\epsilon$ are given in Fig. 5. In the figure, lines I, II and III are for the cases of fixed, $1/\sqrt{\epsilon}$, and uniform RPD respectively.

![Graph showing average laminar lengths versus $\epsilon$ for Type-II intermittency.](image1)

**Fig. 4.** Average laminar lengths versus $\epsilon$ for Type-II intermittency. Lines I–III are for the cases of fixed, of the form $1/\sqrt{\epsilon}$ and uniform RPD respectively. The critical exponents of lines I–III are $-1$, $-3/4$ and $-1/2$ respectively, for $\Delta = 10^{-10}$. 

![Graph showing average laminar lengths versus $\epsilon$ for Type-III intermittency.](image2)

**Fig. 5.** Average laminar lengths versus $\epsilon$ for Type-III intermittency. Lines I–III are for the cases of fixed, of the form $1/\sqrt{\epsilon}$ and uniform RPD respectively. The critical exponents of lines I–III are $-1$, $-3/4$ and $-1/2$ respectively, for $\Delta = 10^{-10}$. 

The solid lines represent the results of analytic calculations, and dots represent numerical results. As is shown in the figure, the laminar lengths of analytic calculation well agree with those of numerical experiment. Both the numerical and analytic calculations show that, if $\epsilon \ll \Delta^{1/2}$, the critical exponents are $-1/2, -3/4$, and $-1$, for the case of uniform, $1/\sqrt{x}$, and fixed RPD respectively, while the critical exponent is always $-1$ if $\epsilon \gg \Delta^{1/2}$.

4. Conclusions

In conclusion the critical exponent is always $-1$ in the limit of $\epsilon \gg \Delta^{1/2}$ as was observed by Pomeau and Manneville. On the other hand, new characteristic relations are emerged on Type-II and Type-III intermittency depending on the RPD, if $\epsilon \ll \Delta^{1/2}$. It is found that, if the RPD is uniform, the critical exponent is $-1/2$; if it is fixed at LBR, $-1$, and finally if it is of the form $1/\sqrt{x}$, $-3/4$. These results show that not only the local structure of Poincaré map, but also the RPD has an essential role in determining the scaling relation in Type-II and III intermittency.

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