

Stochastic Processes and the Dirac Equation with External Fields

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The equation describing the stochastic motion of a classical particle in 1+1-dimensional space-time is connected to the Dirac equation with external gauge fields. The effects of assigning different turning probabilities to the forward and the backward moving particles in time are discussed.

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For many decades, the relations between stochastic processes and quantum mechanics have attracted much attention [1–3]. The Schrödinger's equation for a nonrelativistic electron and the heat equation are related [4,5] as the Feynman path integral [6] and the Wiener integral are connected by an analytic continuation. This correspondence was extended to the relativistic case by studying the telegrapher's equation which produced the Dirac equation when analytically continued [7]. A decade ago, McKeon and Ord further extended the study by introducing the probability of moving backward in time and deriving the Dirac equation in 1+1-dimensional space-time without analytical continuation [8].

In this paper, we will extend these considerations by introducing external gauge fields and giving different 'turning' probabilities to forward and backward moving particles in time. In these models, particles are supposed to suffer random, Poisson-distributed reversals in the moving direction. Notations and procedures similar to those used in Ref. 8 are adopted. Let F_+ (F_-) and B_+ (B_-) be the probabilities for moving to the positive(negative) x -direction for forward-moving particles and backward-moving particles, respectively, in the t -direction. The probability for reversal in time interval Δt is a_R^F (a_L^F) Δt for turning right(left) with respect to the direction of motion for the forward moving particles, and a_R^B (a_L^B) Δt for the backward moving particles(See Fig. 1). In this paper, we assume that $a_{R,L}^{F,B}$ can be analytically continued to the imaginary part. This requires a

non-trivial interpretation against the positiveness of the probability.

When there is an external field, it is reasonable to assume that the forward and the backward moving particles have different turning probabilities, *i.e.*, $a_{R,L}^F \neq a_{R,L}^B$. From Fig. 1, it is easy to derive the master equation describing the evolution of the probability F_{\pm} on

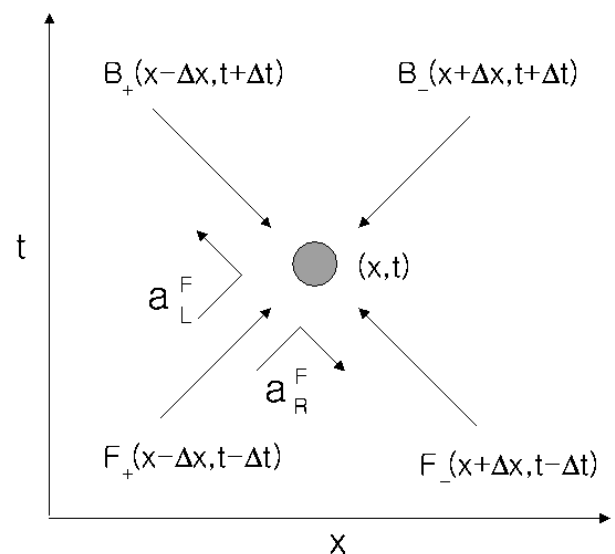


Fig. 1. Schematic diagram showing the stochastic motion of particles in the $x-t$ plane and the definitions of the probabilities. The center dot denotes the point (x, t) .

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the $x - t$ plane:

$$\begin{aligned} F_{\pm}(x, t) &= (1 - a_L^F \Delta t - a_R^F \Delta t) F_{\pm}(x \mp \Delta x, t - \Delta t) \\ &+ a_{L,R}^B B_{\pm} \Delta t (x \mp \Delta x, t + \Delta t) \\ &+ a_{R,L}^F F_{\mp} \Delta t (x \pm \Delta x, t - \Delta t). \end{aligned} \quad (1)$$

Then, the ‘causality condition’ used in Ref. 8 that $F_{\pm}(x, t) = B_{\mp}(x \pm \Delta x, t + \Delta t)$ gives similar equation for B_{\pm} :

$$\begin{aligned} B_{\pm}(x \mp \Delta x, t + \Delta t) &= F_{\mp}(x, t) \\ &= (1 - a_L^F \Delta t - a_R^F \Delta t) B_{\pm}(x, t) \\ &+ a_{R,L}^B \Delta t F_{\pm}(x, t) + a_{L,R}^F \Delta t B_{\mp}(x, t). \end{aligned} \quad (2)$$

Expanding Eq. (1) and Eq. (2) to first order in Δx and Δt leads to two differential equations:

$$\begin{aligned} \pm v \frac{\partial F_{\pm}}{\partial x} + \frac{\partial F_{\pm}}{\partial t} &= -a_{L,R}^F F_{\pm} + a_{L,R}^B B_{\pm} \\ &- a_{R,L}^F F_{\pm} + a_{R,L}^F F_{\pm} \end{aligned} \quad (3)$$

and

$$\begin{aligned} \pm v \frac{\partial B_{\mp}}{\partial x} + \frac{\partial B_{\mp}}{\partial t} &= -a_{L,R}^F B_{\mp} + a_{L,R}^B F_{\mp} \\ &- a_{R,L}^F B_{\mp} + a_{R,L}^F B_{\pm}, \end{aligned} \quad (4)$$

where $v \equiv dx/dt$. Subtracting Eq. (4) from Eq. (3) yields

$$\begin{aligned} v \frac{\partial (F_{\pm} - B_{\mp})}{\partial x} \pm \frac{\partial (F_{\pm} - B_{\mp})}{\partial t} &= \\ &\mp (a_{L,R}^F + a_{R,L}^F)(F_{\pm} - B_{\mp}) \\ &\pm (a_{R,L}^F - a_{L,R}^B)(F_{\mp} - B_{\pm}). \end{aligned} \quad (5)$$

Now, we will show that this equation can be related to the 1+1-dimensional Dirac equation with external fields.

First, let us consider the case where the external fields are gauge fields. The Dirac equation for electrons with $U(1)$ gauge fields is

$$(i\gamma^{\mu}(\hbar\partial_{\mu} - ieA_{\mu}) - mc^2)\psi = 0. \quad (6)$$

As is well known, by a gauge transformation $A_1 \rightarrow A_1 + \partial_1\Lambda$ with a function Λ satisfying $\partial_{\mu}\partial^{\mu}\Lambda = 0$ in the Lorentz gauge $\partial^{\mu}A_{\mu} = 0$, A_1 can be always chosen as zero. Then, in 1 + 1 dimensions, the Dirac equation can be written as

$$i\hbar\partial_t\psi + i\hbar\sigma_z\partial_x\psi = -eA_0\psi + mc^2\sigma_y\psi, \quad (7)$$

where $\gamma^0 = \sigma_y$, $\gamma^1 = i\sigma_x$, and σ_x and σ_y are the Pauli matrices. Substituting a two-dimensional spinor $\psi^T \equiv (\psi_+, \psi_-)$ into Eq. (7) gives

$$c\partial_x\psi_{\pm} \pm \partial_t\psi_{\pm} = \pm \frac{ieA_0}{\hbar}\psi_{\pm} - \frac{mc^2}{\hbar}\psi_{\mp}. \quad (8)$$

Then, using the identities

$$\begin{aligned} v &= c, \\ F_{\pm} - B_{\mp} &= \psi_{\pm}, \\ (a_R^F - a_L^B) &= -(a_L^F - a_R^B) = -mc^2/\hbar, \end{aligned} \quad (9)$$

$$(a_L^F + a_R^F) = -ieA_0/\hbar, \quad (10)$$

one easily finds that Eq. (5) can be reduced to Eq. (8), the 1+1-dimensional Dirac equation with external gauge fields. Note that from Eq. (9), $a_R^F + a_L^F$ should be equal to $a_R^B + a_L^B$ to guarantee a unique mass m for the particle. Obviously, this relation is automatically satisfied when $a_{R,L}^F = a_{R,L}^B \equiv a_{R,L}$ as in Ref. 8, where ψ_{\pm} was defined as $\exp\{(a_L + a_R)t\}(F_{\pm} - B_{\mp})$ rather than $(F_{\pm} - B_{\mp})$ itself. Since the gauge fields contribute to the phase of the matter fields as $\exp\{ie \int dx^{\mu} A_{\mu}/\hbar\}$, it seems to be reasonable to take $a_L + a_R$ to be proportional to A_0 . Furthermore, when $a_{R,L}^F = a_{R,L}^B \equiv a_{R,L}$, one can obtain a_R and a_L explicitly from Eq. (9) and Eq. (10), *i.e.*,

$$\begin{aligned} a_R &= \frac{-ieA_0 - mc^2}{2\hbar}, \\ a_L &= \frac{-ieA_0 + mc^2}{2\hbar}. \end{aligned} \quad (11)$$

If the gauge fields are space-time dependent, the $a_{R,L}^{F,B}$ are not constants, but functions of x, t . However, including this space-time dependency of $a_{R,L}^{F,B}$ in Eq. (9) and Eq. (2), for example, by replacing a_R^F by $a_R^F(x + \Delta x, t - \Delta t)$, does not change the results because it gives the second-order terms when expanded in Δx and Δt . As one can see in Eq. (11), the $a_{R,L}^{F,B}$ are complex numbers when the external fields are gauge fields. This means that we need an analytic continuation, which was already noted in Ref. 8. This is unavoidable because the terms $\hbar\partial_{\mu} - ieA_{\mu}$ in the Dirac equation always give a factor of i for A_{μ} in comparison with ∂_{μ} .

In summary, the relation between the stochastic motion and the Dirac equation with external gauge fields in 1+1-dimensional space-time is studied. We also investigated the effects of assigning different turning probabilities to the particles moving forward and backward in time.

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